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Test 2 December 8, 2008

1. The E/M field strength tensor is

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

and $\epsilon^{\alpha\beta\gamma\delta}$ is Levi-Civita tensor. Find the dual field strength tensor $\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$.

Solution. The properties to use here are (i) all elements of the Levi-Civita tensor where two of the four indices are the same are 0, and (ii)the number of transpositions it takes to get the indexing into $0, 1, 2, 3$ format

$$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta} = 2 \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

2. If the electric field $\vec{E}$ is perpendicular to magnetic field $\vec{B}$ in a frame, show the electric field is perpendicular to the magnetic field in all frames.

Solution. Make use of the fact that the quantity $\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = 8\vec{E}\cdot\vec{B}$ is scalar. Another way to go about this was done in the homework.

Alternate Solution

Assume WLOG that boost is in x-direction and use the definitions for β and γ given above.

$$\begin{aligned} \mathbf{E}'\cdot\mathbf{B}' &= E_xB_x + \gamma^2[(E_y - \beta B_z)(B_y + \beta E_z) + (E_z + \beta B_y)(B_z - \beta E_z)] \\ &= E_xB_x + \gamma^2[E_yB_y + \beta E_yE_z - \beta B_yB_z - \beta^2 E_zB_z \\ &+ E_zB_z - \beta E_yE_z + \beta B_yB_z - \beta^2 E_yB_y] \\ &= E_xB_x + E_yB_y + E_zB_z \\ &= \mathbf{E}\cdot\mathbf{B}. \end{aligned}$$

So, $\mathbf{E}\cdot\mathbf{B}$ is invariant wrt to Lorentz transformation.

3. A particle of mass m and charge q is accelerated by a uniform electric field E from time $t = 0$ to t. The initial velocity of the particle is zero, find the velocity of the particle at t.

Solution. Assume electric field is in x-direction. Particle is initially at rest, so $p_x = p_y = p_z = 0$. Additionally, $\dot{p}_y = \dot{p}_z = 0$. For the x-direction, $\dot{p}_x = qE$. So, the momentum integrates to

$$p_x = \int \dot{p}_x dt = qEt + C = qEt \quad \text{by initial conditions.}$$

Since particle is initially at rest, $\mathrm{E}_0 = mc^2$. The energy of the particle evolves as

$$\begin{aligned} \mathrm{E} &= \sqrt{m^2c^4 + c^2(qEt)^2} \\ &= \sqrt{\mathrm{E}_0^2 + c^2(qEt)^2}. \end{aligned}$$

Take relativistic momentum into account

$$\begin{aligned} p_x = mc\frac{dU}{ds} &= \frac{1}{c^2}\left(\frac{mc^2}{\sqrt{1 - v^2/c^2}}\right)\frac{dx}{dt} \\ \implies \quad \left.\frac{dx}{dt}\right|_{t=T} = v(T) &= \frac{c^2 p_x}{\mathrm{E}_0} \\ &= \frac{c^2 qET}{\sqrt{m^2c^4 + (cqET)^2}}. \end{aligned}$$

4. Consider a static magnetic field $\vec{B}(z)$, which is mainly in the z-direction, and the intensity $B(z)$ is slowly varying with z. Find the adiabatic invariance of a charged particle moving in this field.

Solution. Start with the formula for the action, see Jackson pg. 593 for more detail

$$I = \frac{1}{2\pi}\oint \vec{P} \cdot dq$$

where the P and q are generalized coordinates in the Hamiltonian sense. $\vec{P} = \vec{p} + \frac{q}{c}\vec{A}$. The particle momentum $\vec{p} = \gamma m \vec{v}_\perp$, where $|\vec{v}_\perp| = a\omega$ and $\omega = \frac{qB}{mc\gamma}$. Now return to the integral,

$$\begin{aligned} I &= \frac{1}{2\pi}\left[\oint \vec{p} \cdot d\vec{l} + \frac{q}{c}\oint \vec{A} \cdot d\vec{l}\right] \\ &= \frac{1}{2\pi}\left[\oint m\gamma\vec{v}_\perp \cdot d\vec{l} + \frac{q}{c}\oint \vec{A} \cdot d\vec{l}\right] \\ &= \frac{1}{2\pi}\left[2\pi\gamma\omega a^2 - \frac{q}{c}B\pi a^2\right] \\ &= \frac{1}{2}Ba^2. \end{aligned}$$